SOLUTION OF SOME MIXED HEAT-CONDUCTION BOUNDARY-VALUE PROBLEMS

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Some mixed boundary-value problems for steady-state heat conduction in a rectangular domain with a variable heat-transfer coefficient are solved by reduction to infinite systems.

A condition of the third kind

$$\frac{\partial u}{\partial n}+h(u-f)=0, \tag{1}$$

corresponding to Newton's-law heat exchange with the ambient medium is often used as a boundary condition in the theory of heat conduction.

Problems are usually solved in the theory for h = const [1, 2]. There are problems of practical interest, however, for which the heat-transfer coefficient h is a function that varies along the boundary. We shall give solutions for a certain class of such problems.

We consider a steady-state distribution of temperature in a cylindrical heat-evolving element of rectangular cross section having distributed heat sources of constant power.

The two lateral surfaces of the element are assumed to be thermally insulated; there is Newton's-law heat transfer to the ambient medium through the other two surfaces and the heat-transfer coefficient varies along these surfaces. Such heat-transfer conditions are realized, in particular, when parts of the lateral surface are cooled by a moving liquid or gas.

With the problem formulated in this way we can also consider the case in which heat is removed through nonideal thermal contacts of arbitrary dimensions located on the surface of the heat-evolving element.

To make the problem general we also introduce bulk heat absorption that is proportional to the temperature of the body at the given point. Thanks to such heat "sinks" the processes associated with the effects of radiation, ionization, etc. can be simulated in a linear approximation. Moreover such a treatment also enables us to investigate the steady-state temperature distribution in a rectangular plate of fairly small height within the framework of the two-dimensional theory; this is done by replacing the heat fluxes through the end surfaces by a certain additional heat absorption introduced into the two-dimensional heat-conduction equation [1].

In accordance with the physical situation we formulate the following mixed boundary-value problem of steady-state heat conduction in the domain $G\{0 \le x \le l,\ -1 \le y \le 1\}$ bounded by contour Γ (Fig. 1):

$$\Delta u - q^2 u = -P,$$

$$\frac{\partial u}{\partial n}+h(u-f)=0 \text{ on } \gamma, \tag{2}$$

$$\frac{\partial u}{\partial n}=0 \text{ on } \Gamma\setminus\gamma.$$

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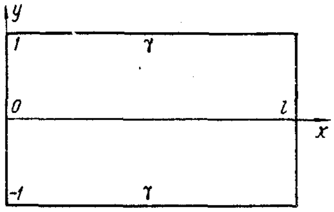

Fig. 1. Domain of variation of the variables.

Here h and f are arbitrary piecewise-smooth functions specified on the boundary γ ($h(x) \geq 0$); P = const > 0 and q^2 = const > 0 are parameters characterizing the power of bulk evolution and absorption of heat, respectively; n is the exterior normal to Γ. Since (2) is linear in P, we shall henceforth take P = 1 with no loss of generality.

We shall confine the discussion to the search for a solution of (2) that satisfies the condition of symmetry about y = 0.

Using the substitution

$$u(x,\ y) = w(y) + v(x,\ y) \equiv \frac{1}{q^2}\left(1 - \frac{\operatorname{ch} qy}{\operatorname{ch} q}\right) + v(x,\ y) \tag{3}$$

we reduce the boundary-value problem (2) to solution of the homogeneous equation

$$\Delta v - q^2 v = 0 \tag{4}$$

with the boundary conditions

$$\frac{\partial v}{\partial x} = 0 \quad \text{for} \quad x = 0 \quad \text{and} = l, \tag{5}$$

$$\pm\frac{\partial v}{\partial y} + h(x)\, v = F(x) \quad \text{for} \quad y = \pm 1, \tag{6}$$

$$F(x) = \frac{\operatorname{th} q}{q} + h(x) f(x).$$

A solution of the form

$$v(x,\ y) = \sum_{k=0}^{\infty} v_k(x,\ y) = \sum_{k=0}^{\infty} a_k \frac{\operatorname{ch} \omega_k y}{\operatorname{ch} \omega_k} \cos \lambda_k x,$$

$$\lambda_k = \frac{\pi k}{l}, \quad \omega_k = (\lambda_k^2 + q^2)^{1/2}. \tag{7}$$

satisfies (4) and (5). The coefficient a_k of this expansion must be found from the boundary condition (6), which leads to the relationship

$$\sum_{k=0}^{\infty} a_k \rho_k \cos \lambda_k x + h(x) \sum_{k=0}^{\infty} a_k \cos \lambda_k x = F(x),$$

$$\rho_k = \omega_k \operatorname{th} \omega_k. \tag{8}$$

On the interval $0 \leq x \leq l$ we represent h(x) and F(x) as the following Fourier series:

$$h(x) = \frac{h_0}{2} + \sum_{k=1}^{\infty} h_k \cos \lambda_k x,$$

$$F(x) = \frac{\nu_0}{2} + \sum_{k=1}^{\infty} \nu_k \cos \lambda_k x. \tag{9}$$

Next we consider the function $\varphi(x) = v(x,\ y)|_{y=1}$, $0 \leq x \leq l$; on the basis of (7), its Fourier series has the form

$$\varphi(x) = \sum_{k=0}^{\infty} a_k \cos \lambda_k x,$$

and we then write the Fourier series for the product of the functions h(x) and $\varphi(x)$:

$$h(x)\varphi(x) = \frac{\theta_0}{2} + \sum_{k=1}^{\infty} \theta_k \cos \lambda_k x, \quad 0 \leqslant x \leqslant l. \tag{10}$$

If h(x) and $\varphi(x)$ belong to the class $L_2(0,\ l)$ of quadratically integrable functions, we can write the following formulas for the coefficients θ_k of expansion (10) [3]:

$$\theta_0 = a_0 h_0 + \sum_{m=1}^{\infty} a_m h_m,$$
$$\theta_k = a_0 h_k + \frac{1}{2}\sum_{m=1}^{\infty} a_m (h_{k-m} + h_{k+m}), \; k = 1, 2, \ldots, \tag{11}$$

where we must take $h_{-k} = h_k$ in (11).

Substituting the expansions (9) and (10) for F(x) and $h(x)\varphi(x)$ into (8) we obtain

$$\sum_{k=0}^{\infty} a_k \rho_k \cos\lambda_k x + \frac{\theta_0}{2} + \sum_{k=1}^{\infty} \theta_k \cos\lambda_k x = \frac{\nu_0}{2} + \sum_{k=1}^{\infty} \nu_k \cos\lambda_k x.$$

Allowing for (11), we then have

$$a_0\left(\rho_0 + \frac{h_0}{2}\right) + \frac{1}{2}\sum_{m=1}^{\infty} a_m h_m = \frac{\nu_0}{2},$$
$$a_k\rho_k + a_0 h_k + \frac{1}{2}\sum_{m=1}^{\infty} a_m (h_{k-m} + h_{k+m}) = \nu_k, \; k = 1, 2, \ldots \tag{12}$$

Eliminating a_0 from (12), we arrive at an infinite system of linear algebraic equations for the unknown coefficients a_k:

$$a_k + \sum_{m=1}^{\infty} C_{km} a_m = \eta_k, \; k = 1, 2, \ldots$$
$$C_{km} = \frac{1}{\rho_k}\left[\frac{1}{2}(h_{k-m} + h_{k+m}) - \frac{h_k h_m}{2\rho_0 + h_0}\right], \tag{13}$$
$$\eta_k = \frac{1}{\rho_k}\left(\nu_k - \frac{\nu_0}{2\rho_0 + h_0} h_k\right).$$

Let us investigate the problem of determining the sequence a_k from the infinite system (13). We note that since $h(x) \geq 0$, then $h_0 > 0$, $|h_k| < 2\rho_0 + h_0$ and therefore

$$|C_{km}|^2 \leqslant \frac{2}{\rho_k}(|h_{k-m}|^2 + |h_{k+m}|^2 + |h_m|^2).$$

Let us show that the matrix (C_{km}) satisfies the condition

$$\sum_{k,\,m=1}^{\infty} |C_{km}|^2 < \infty. \tag{14}$$

In fact,

$$\sum_{k,\,m=1}^{\infty} |C_{km}|^2 \leqslant \sum_{k=1}^{\infty} \frac{2}{\rho_k} \sum_{m=1}^{\infty} (h_{k-m}^2 + h_{k+m}^2 + h_m^2) =$$
$$= \sum_{k=1}^{\infty} \frac{2}{\rho_k}\left(\sum_{m=-\infty}^{k-1} h_m^2 + \sum_{m=k+1}^{\infty} h_m^2 + \sum_{m=1}^{\infty} h_m^2\right) \leqslant \tag{15}$$
$$\leqslant \text{const} \sum_{k=1}^{\infty}\left(\frac{1}{k^2}\sum_{m=0}^{\infty} h_m^2\right).$$

Since h_k are Fourier coefficients, the series $\sum_{m=0}^{\infty} h_m^2$ converges so that the validity of the estimate (14) follows from (15).

The inequality

$$\sum_{k=1}^{\infty} |\eta_k|^2 < \infty \tag{16}$$

is valid for the sequence of free terms η_k of the infinite system (13).

Thanks to the estimates (14) and (16) that we have found, we can use a reduction method to find the coefficients a_k from (13) [4]; the method consists in the following: we take as an approximate solution of

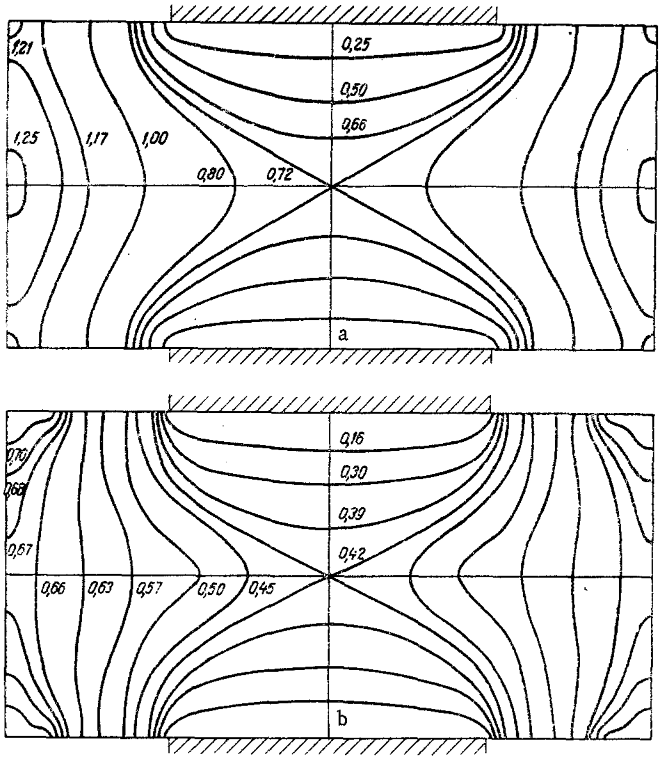

Fig. 2. Lines of isotherms for q = 0 (a) and q = 1 (b).

the infinite system (13) the solution of the corresponding abridged system

$$a_k^* + \sum_{m=1}^{N} C_{km} a_m^* = \eta_k, \quad k = 1,\ 2,\ \ldots,\ N,$$

where we have convergence of the approximate solution a_k^* to the exact a_k when $N \to \infty$.

On the basis of the first relationship of (12) the value of a_0^* should be determined from the formula

$$a_0^* = \frac{1}{2\rho_0 + h_0}\left(\nu_0 - \sum_{m=1}^{N} a_m^* h_m\right).$$

Thus the solution of the initial boundary-value problem (2) is represented by the expression

$$u(x,\ y) = \frac{1}{q^2}\left(1 - \frac{\operatorname{ch} qy}{\operatorname{ch} q}\right) + \sum_{k=0}^{\infty} a_k \frac{\operatorname{ch} \omega_k y}{\operatorname{ch} \omega_k} \cos \lambda_k x \qquad (17)$$

with coefficients a_k satisfying (13), which can be solved by reduction.

For the special case in which $h(x) = H = \text{const}$ in the interval $0 \le x \le l$ the matrix (C_{km}) becomes a diagonal matrix, the infinite system (13) splits up and the exact values of the coefficients in (17) can be written out:

$$a_0 = \frac{\nu_0}{2(\rho_0 + H)}, \quad a_k = \frac{\nu_k}{\rho_k + H}.$$

The relationships (13) and (17) that we have obtained were used to calculate the temperature distribution in problems with various h(x) dependences. In particular, it is interesting to consider the case in which h(x) is a continuous function

$$h(x) = \begin{cases} 0 & \text{for} \quad 0 \leqslant x < x_0,\ l - x_0 < x \leqslant l, \\ H_0 = \text{const} & \text{for} \quad x_0 < x < l - x_0. \end{cases}$$

Specification of h(x) in this way corresponds to transfer of heat to the external medium through nonideal thermal contacts having arbitrary dimensions.

Figure 2a, b shows the temperatures in such problems for cases q = 0 and q = 1, respectively, when $f = 0$, $H_0 = 100$, $i = 4$, and $x_0 = 1$. For clarity the boundary segments corresponding to the thermal contacts are shown hatched in the figures.

To conclude, we note that an analogous method may be used to solve the initial problem with a boundary condition of the first kind on $\Gamma\setminus\gamma$ the solution of such a problem is also represented as a trigonometric series with coefficients determined from the corresponding infinite system.

NOTATION

u	is the temperature;
h	is the heat-transfer coefficient;
f	is the temperature of the external medium;
x, y	are the coordinates;
l	is a parameter characterizing the ratio of the sides of the rectangle;
a_k, h_k, ν_k, θ_k	are the Fourier coefficients;
ρ_k, ω_k, λ_k, η_k	are the parameters that depend on the summation index;
C_{km}	is the matrix elements of the infinite system;
F, v, w	are the auxiliary functions.

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